# Memory efficient algorithm for solving the inverse gravimetry problem of finding several boundary surfaces in multilayered medium

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# Memory Efficient Algorithm for Solving the Inverse Gravimetry Problem of Finding Several Boundary Surfaces in Multilayered Medium

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**Abstract.** For solving the inverse gravimetry problem of finding several boundary surfaces in a multilayered medium, the parallel algorithm was constructed and implemented for multicore CPU using OpenMP technology. The algorithm is based on the modified nonlinear conjugate gradient method with weighting factors previously proposed by authors. To reduce the memory requirements and computation time, the modification was constructed on the basis of utilizing the Toeplitz-block-Toeplitz structure of the Jacobian matrix of the integral operator. The model problem of reconstructing three surfaces using the quasi-real gravitational data was solved on a large grid. It was shown that the proposed implementation reduces the computation time by 80% in comparison with the earlier algorithm based on calculating the entire matrix. The parallel algorithm shows good scaling of 94% on 8-core processor.
Keywords: Gravimetry problem, nonlinear gradient methods, Toeplitz matrix, parallel algorithms
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## INTRODUCTION

The problem considered in this paper consists in finding several interfaces between layers in a multilayered medium using known gravitational data [1, 2, 3]. This problem is described by a nonlinear integral equation of the first kind; so, it is ill-posed. The real gravitational measurements are carried out over a large area with the large-scale grids. Processing the gravitational data is a time consuming process that requires a lot of memory. So, it is necessary to develop parallel algorithms for parallel computing systems.

In works [4, 5], for solving the problem in the case of a single surface, the authors have proposed an algorithm based on utilizing the Toeplitz-block-Toeplitz structure of the Jacobian matrix of the integral operator.

Here, we construct a parallel algorithm on the basis of the nonlinear conjugate method utilizing the structure of the Jacobian matrix for the case of multiple surfaces and implement it on multicore CPU using the OpenMP technology.

We compare new algorithm with the earlier variants in terms of computation time in solving the model problem with quasi-real gravitational data.

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## STATEMENT OF THE INVERSE GRAVIMETRY PROBLEM FOR MODEL OF MULTILAYERED MEDIUM

We assume that the lower half-space is composed of several layers with constant densities, which are separated by the sought surfaces $S_l,\ l = 1, ..., L$, where $L$ is the number of boundary surfaces.

The gravitational field generated by this half-space is equal to the sum of the gravitational fields due to each surface. Let the boundary surfaces be specified by the functions $z = \zeta_l(x, y)$, let the density contrasts on them be $\Delta\sigma_l$, and let the surfaces have the horizontal asymptotic planes $z = H_l$.

The field $\Delta g$ (produced by the superposition of the boundary surfaces and measured on the Earths surface $z = 0$) is described by the following equation (with accuracy up to a constant term of summation) [1]:

$$f \sum_{l=1..L} \Delta\sigma_l \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big( \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + \zeta^2(x',y')}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + \zeta^2(x',y')}} \Big) dx'dy' = \Delta g(x, y, 0), \tag{1}$$

where $f$ is the gravitational constant.

This equation is the Fredholm nonlinear equation of the first kind of functions $\zeta_l$ and is ill-posed; it has a nonunique solution, which unstably depends on the initial data.

After discretization of equation (1) on the rectangular grid $n = M \times N$, where the right-hand side $\Delta g(x, y, 0)$ is given, and approximation of the left-side integral operator $A(\zeta_1, ..., \zeta_l)$ by the quadrature rules, we obtain: the vector of the right-hand side $F$ with the length $n$, the combined surfaces vector $z = [\zeta_1(x_1, y_1), ..., \zeta_1(x_M, y_N), ..., \zeta_2(x_1, y_1), ..., \zeta_2(x_M, y_N), ..., \zeta_L(x_M, y_N)]$ with the length $Ln$, and the system of nonlinear equations having the form

$$A(z) = F. \tag{2}$$

## ALGORITHM FOR SOLVING THE INVERSE PROBLEM

In this work, to solve problem (2), we use the modified linearized conjugate gradient method with weighting factors [2]. The main idea of modification is calculating the Jacobian matrix $A'(z^k)$ of the nonlinear operator $A$ at an initial point $z^0$ without updating it in the entire iterative process.

This resulting method has the following form:

$$\begin{gathered} z^{k+1} = z^k - \psi \frac{\langle p^k, S_0(z^k)\rangle}{\left\|A'(z^0) - p^k\right\|} p^k, \quad p^k = v^k + \beta^k p^{k-1}, \quad p^0 = v^0, \\ \beta^k = \max\left\{ \frac{\langle v^k, v^k - v^{k-1}\rangle}{\left\|v^{k-1}\right\|}, 0 \right\}, \quad v = \gamma \circ S(z), \quad S_0(z) = A'(z^0)^*(A(z) - F), \end{gathered} \tag{3}$$

where $z^k$ is the solution estimate at the $k$th iteration, $\psi$ is the damping factor, $\gamma$ is the vector of weighting factors, $A'(z)$ is the Jacobean matrix of the discretized integral operator A(z), $\circ$ is the operation of componentwise vector multiplication.

In this work, we propose the following rules for choosing the weighting factors:

$$\begin{gathered} [F_1, ..., F_L] \to [f_1, ..., f_{Ln}] \to [\gamma_1, ..., \gamma_{Ln}], \quad F_l \to [\gamma_{n(l-1)}, \gamma_{n(l-1)+1}, ..., \gamma_{nl}], \\ \gamma_l = \frac{\sqrt{f_i^2 + \mu}}{\max\limits_{i=1}^{n}\left\{ \sqrt{f_i^2 + \mu} \right\}}, 0 < \mu < 1, \end{gathered} \tag{4}$$

where $\mu$ is the smoothing parameter.

The fields $F_l$ are extracted from the total field $F$ using the heightwise transformation technique from [6].

The condition $\|A(z) - F\| / \|F\| < \varepsilon$ for sufficiently small $\varepsilon$ is used as the termination criterion for the iterative process.

## STRUCTURE OF THE JACOBIAN MATRIX

Let us consider the structure of the Jacobian matrix. For the modified methods, we use the fixed point $z^0$. The elements of the matrix $A'(z^0)$ will have the following form

$$\begin{aligned} &a_{i,j,u,v,l} = a_{(j-1)M+i+lMN,(v-1)M+u} = f\Delta\sigma_l \Delta x \Delta y \frac{-z^0}{((x_u - x_i)^2 + (y_v - y_j)^2 + (z^0)^2)^{3/2}}, \\ &i, u = 1, ..., M, \quad j, v = 1, ..., N, \quad l = 1, ..., L. \end{aligned} \tag{5}$$

The scheme of matrix structure is shown in Fig. 1. On top level, the matrix consists of the square blocks of the sise $MN \times MN$, where each block corresponds to one of the sought surfaces. Inside each block, the row consists of partial derivatives by all integration points for one fixed observation point. These blocks have subblocks of $M \times M$ size, and consist of $N \times N$ subblocks. Each element of the matrix have five indices: $l$ defines the big square block,$(j, v)$ define the subblock, $(i, u)$ define the element inside the subblock.

Note that the values of elements depends only on their indices. Each big square block is a Toeplitz-block-Toeplitz matrix, *i.e.*, in each subblock, each diagonal descending from left to right consists of the same element, and each subblock diagonal consists of the same subblock. Thus, to store the matrix $A'(z^0)$, we can use the same technique as we used in work [5]. For and $MN \times LMN$ matrix, we need to store only $L(2M - 1)(2N - 1)$ unique elements. For example, for $512 \times 512$ grid, we need 1536 GB for full matrix storage or 25 MB for compressed storage (65536 times lower).

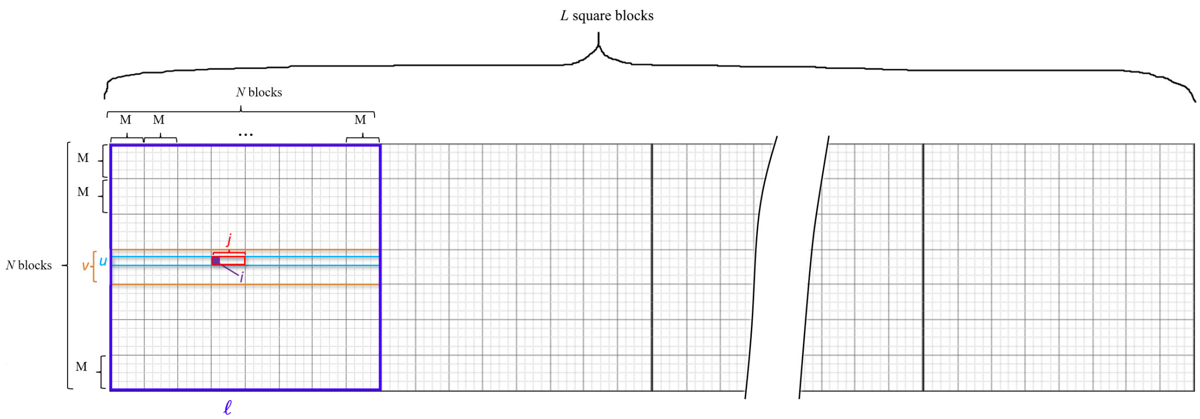


**FIGURE 1.** Scheme of matrix structure

## PARALLEL IMPLEMENTATION AND NUMERICAL EXPERIMENT

For solving the inverse problem, the parallel algorithm was developed for multicore processor using OpenMP technology.

Most expensive part is calculation of the left-hand estimate $A(z^k)$ at each iteration. The vector are divided into a number of fragments, and each fragment is processed separately by its own thread.

To test the constructed algorithm and to compare it with the unmodified one in terms of execution time, we use the model problem of reconstructing three boundary surfaces [3] on a large grid ($512 \times 512$ nodes) using the quasi-real data.

The model gravitational field was obtained by solving the forward problem using three defined surfaces $\zeta_1, \zeta_2, \zeta_3$ with asymprotic planes $H_1 = 10$ km, $H_2 = 20$ km, $H_3 = 30$ km and density contrasts $\Delta\sigma_1 = \Delta\sigma_2 = \Delta\sigma_3 = 0.2$ g/cm$^3$.

These surfaces were constructed on the basis of gravity maps [7] for an area of $600 \times 600$ km near Ekaterinburg, Russia.

The inverse problem was solved using the eight-core Intel Xeon E5-2650 processor incorporated into the Uran parallel computing system. The Jacobian matrix for this problem is $262144 \times 786432$. The smoothing parameter was $\mu = 0.2$. For the stopping criterion, $\varepsilon = 0.1$ was used. The relative errors $\left\|\widetilde{\zeta_l} - \zeta_l\right\| / \|\zeta_l\|$ of reconstructed surfaces $\widetilde{\zeta_1}, \widetilde{\zeta_2}, \widetilde{\zeta_3}$ are lower than 1%.

Table 1 contains the execution times $T_1$ and $T_8$ on one and eight cores of Xeon E5-2650 CPU, number of iterations $N$, speedup $S_m = T_1/T_m$ and efficiency $E_m = S_m/m$ for two algorithms ($m = 8$):

- calculating the entire matrix without exploiting its structure;
- using the Toeplitz-block-Toeplitz structure of the Jacobian matrix.

**TABLE 1.** Results of the numerical experiments.

| Algorithm | $T_1$, hours | $T_8$, hours | $N$ | $S_8$ | $E_8$ |
|---|---|---|---|---|---|
| Entire matrix | 14.7 | 2.9 | 12 | 6.1 | 0.77 |
| Toeplitz-block-Toeplitz matrix | 5 | 0.66 | 11 | 7.5 | 0.94 |

## CONCLUSION

For solving the inverse gravimetry problem of finding several boundary surfaces in a multilayered medium, the parallel algorithm was constructed and implemented for multicore CPU using OpenMP technology. The algorithm is based on the modified nonlinear conjugate gradient method with weighting factors previously proposed by authors. To reduce the memory requirements and computation time, the modification was constructed on the basis of utilizing the Toeplitz-block-Toeplitz structure of the Jacobian matrix of the integral operator. As a result, the new algorithm requires 65536 times lower than the earlier implementations.

The model problem of reconstructing three surfaces using the quasi-real gravitational data was solved on a large grid. It was shown that the proposed implementation reduces the computation time by 80% in comparison with the earlier algorithm based on calculating the entire matrix. The parallel algorithm shows good scaling of 94% on 8-core processor.

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